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AN APPLICATION OF THE METHOD OF LINES TO THE TRANSONIC AIRFOIL PROBLEM

John Richard McCracken

Air Force Institute of Technology
Wright-Patterson Air Force Base, Ohio

December 1971

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13. ABSTRACT

This study presents the development and evaluation of a numerical solution to the transonic airfoil problem. For this initial investigation of the solution method the scope is restricted to symmetrical airfoils at zero incidence angle in an inviscid flow field. The small perturbation relationship and the irrotationality condition are selected for the set of governing equations and after limiting their domains by a coordinate transformation, the set is reduced to a system of ordinary differential equations by the method of lines. Solutions to this system are then evaluated by comparison against experimental data on a subcritical, critical, and supercritical airfoil and against the exact solutions to the subsonic and supersonic infinite wavy wall. It is concluded that the proposed solution technique, if given the correct initial conditions, can produce extremely accurate results with very short computation time and storage space.

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Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

Air University

in Partial Fulfillment of the

Requirements for the Degree of

Master of Science

by

John R. McCracken, B.S.A.E.

2/Lt USAF

Graduate Aerospace-Mechanical Engineering

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Preface

There are several people who contributed to this study and I'd like to take this opportunity to recognize a few of them. Of course the most substantial contribution was made by my thesis advisor Capt Stephen J. Koob who originally suggested this study and was a constant source of knowledge and encouragement. In addition, special thanks are in line to Lt Col James L. Thompson and Capt John V. Kitowski for their freely given technical advice and to Mrs. Jane Manemann for her efficiency in typing and her aid in assembling this report. Finally, the deepest thanks must go to my wife Jocelyne and son John who have made innumerable personal sacrifices not only through the period of this study but throughout my entire education.

John R. McCracken

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List of Symbols

a_i	coefficients of the series approximation for u
b_j	coefficients of the series approximation for v
c	chord length of the airfoil
c_k	coefficients of the series approximation for v
C_p	coefficient of pressure
f	body shape function
L	arbitrary positive constant
M_l	local Mach number
M_∞	freestream Mach number
U_∞	freestream uniform velocity
u	perturbation velocity in the x-direction
V_l	local velocity
v	perturbation velocity in the y-direction
η	transformation coordinate for y
γ	specific heat ratio
Ω	dummy variable used for the nonlinear algebraic relation
ξ	transformation coordinate for x

AN APPLICATION OF THE METHOD OF LINES TO THE TRANSONIC AIRFOIL PROBLEM

I. Introduction

Background

The demands for faster and more efficient subsonic aircraft have forced designers to reconsider the design problems associated with transonic flow. Until recently, the complications inherent in this flight realm could be avoided by designing the aircraft for its optimum performance either below or above the transonic regime. With this approach the surrounding flow field could be considered as being either entirely subsonic or completely supersonic; however, these simplified approaches are impossible in the design of the aircraft presently being asked for. Cargo and passenger planes that can fly at very high subsonic speeds and interceptors required to fly at these same speeds to most effectively deliver their weapons will want their optimum performance in the transonic portion of the velocity spectrum. Therefore, solutions to the long standing transonic flow problem are now a necessity and are of prime importance in this the initial stage of research and development.

The first and perhaps most difficult hurdle that must be crossed in transonic design problems is the development of a solution technique for some mathematical model of the flow field about a 2-dimensional airfoil in transonic flight. There are several reasons why aerodynamicists have found this

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to be difficult and have avoided it whenever possible; all of the reasons are related to what is physically occuring to the flow in the region of the airfoil. By definition, the transonic airfoil problem is concerned with determining the pressure distribution over an airfoil when the flow field around it is a mixture of subsonic ($M < 1$) and supersonic ($M > 1$) flow regions. When these regions coexist in a field it is referred to as a supercritical flow while a subcritical flow is defined to be a flow in which the velocity is subsonic at every point and a critical flow refers to flow when $M = 1$ at only one point in the field. There are two important mathematical complications that arise when the flow is supercritical. First, if the third independent variable, namely time, is not retained in the differential equations, then the basic form of the equations changes from elliptic in subsonic regions to hyperbolic in supersonic regions. Second, for both the steady and unsteady problem, as the freestream Mach number approaches unity, nonlinear terms must be retained in the small perturbation equations often used as a mathematical model. In addition, a third major mathematical difficulty appears in attempting to model the effects of the shock wave that is normally attached to the airfoil surface. To account for this shock either a complicated system of conservation equations in divergence form must be used or additional relationships (Rankine-Hugoniot equations) must be included and thus amplify the complexity of the problem. With all these added difficulties induced

by the physical nature of the transonic flow field, the mathematical problem has defied exact solution except for extremely limited cases. Fortunately, though, the development and availability of high speed computers has produced approximate solutions by numerical techniques which in some cases are in better agreement with flight data than wind tunnel data.

Statement of the Problem

The objective of this work is to develop and evaluate a particular numerical solution technique for the two-dimensional, inviscid transonic airfoil problem. The method is required to serve as an efficient and workable design tool for the evaluation of ar airfoil in its transonic regime; therefore, it must yield acceptable results with minimum computer time and storage space. As presented here the method is applied only to thin, symmetrical airfoils with sharp leading edges. This permits the present study to concentrate on developing and testing the solution technique and then, if warranted, future efforts could be concentrated on extending the technique to problems of more extensive scope. Nothing will be included to explicitly account for the presence of shock waves in the flow, and the associated viscous interaction with the boundary layer will not be considered. This last assumption, though, is made simply because it is required to reduce the mathematical problem to the scope of this presentation; the intent is not to

imply that inviscid flow is always a good assumption for this velocity regime. Pearcy (Ref 9) has shown that the complicated interaction between the shock wave and the boundary layer will result in flow separation and disturbances not present in flow fields outside the transonic regime. Therefore, it is agreed that these viscous effects could indeed be a dominant feature for many practical problems; however, at this time there is no practical way to approach the full viscous problem within the scope proposed here. It is hoped that the solution technique developed in this presentation may be extendable to the viscous problem but the method as presented here will only be applicable to transonic airfoil problems where the viscous effects do not drastically alter the flow field.

<u>Other Solutions Currently Available</u>

The approach to this problem was selected after examining existing solution methods. As implied before, exact solution techniques for the inviscid transonic airfoil problem are available; however, they contain well-nigh insurmountable difficulties except for extremely limited cases as shown by Ferrari and Tricomi (Ref 2:562). Because of the lack of exact solutions many numerical methods have been devised to produce approximate solutions. These, in general, may be classified into one of three broad categories. In the first category, the governing equations are transformed into integral relations and solved by iterative

procedures. This method has been employed by Oswatitsch (Ref 2:563-574) and more recently by Spreiter and Alksne (Ref 10) in solving the transonic problem for thin, sharp-nosed airfoils. Although this technique could satisfy the computation time objective, at the moment it is not extendable to arbitrary airfoil shapes. The second category of solutions transforms the differential equations into algebraic expressions by an explicit or implicit finite-difference scheme. Magnus and Yoshihara (Ref 7) have been particularly successful with this technique and can now produce extremely good results for several airfoil shapes. The difficulty associated with these methods is the long computer times (on the order of 1 hour/input for the Magnus and Yoshihara method) and storage space required. This makes the method unattractive as a design tool and infeasible to extend to the more involved problems. The third category of solution techniques reduces the system of governing partial differential equations to a set of ordinary differential equations which may be solved by one of several available solution procedures. Dorodnitsyn (Ref 1) originally suggested this approach to the transonic problem, and the recent successful application of these methods by Tai (Ref 11) and Melnik and Ives (Ref 8) prompted the decision to use a version of this technique for the development of the proposed method. By taking a very simplified method of weighted residuals approach, Tai has shown that this method is not only applicable to the transonic airfoil problem, but also that it can

produce excellent results with very little computer time. Meanwhile, Melnik and Ives have used a similar approach and have shown that weighting functions of the subdomain type are capable of handling any arbitrary airfoil shape if the flow field is of finite domain.

Approach to the Problem

Small perturbation theory and irrotationality will be used to provide the set of governing differential equations. The two boundary conditions these equations will be subject to are: no flow normal to the airfoil surface and undisturbed flow at infinity. Prior to developing the solution technique, the form of the governing equations will be changed by nondimensionalizing the variables and making a coordinate transformation to reduce the domain to a finite region. This is done to simplify application of the numerical technique and to eliminate the difficulty associated with applying the boundary condition at infinity.

The particular solution technique to be applied is a form of the method of lines to reduce the set of governing equations to a set of ordinary first-order differential equations. This new set is then solved by the classical Runge-Kutta fourth-order technique for the perturbation velocities throughout the field. Finally, the solution technique is evaluated by comparing the analytical results with published experimental data for symmetrical airfoils at zero incidence angle.

II. Formulation of the Solution Technique

In developing an approach to the transonic airfoil problem three very basic but crucial decisions must be made. First, a set of governing equations must be selected which completely define the problem within some prescribed scope. Next, modifications of the set, such as coordinate or variable transformations must be considered if they will enhance the solution or simplify the mechanics of obtaining the solution. Finally, from the variety of available numerical schemes, one must be selected to solve the resulting system of equations. The product of these three decisions defines a specific solution technique; therefore, the alternatives to each of these choices will be discussed in explaining the rationale for the approach used in this presentation.

Picking a set of governing equations for a solution method requires a choice to be made based on the exactness demanded of the solution versus the amount of mathematical complexity acceptable in the method. For inviscid flow there are three different sets of relationships which are commonly used to define the transonic airfoil problem and the complexity of each of these sets is directly related to how the shock waves are mathematically accounted for. The first set consists of the isentropic relationship

$$\frac{P}{P_o} = \frac{\rho}{\rho_o}^{\gamma}$$

and the unsteady form of the conservation equations for mass and momentum

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} = 0$$

$$\rho \frac{\partial U}{\partial t} + \rho U \frac{\partial U}{\partial x} + \rho V \frac{\partial U}{\partial y} = - \frac{\partial P}{\partial x}$$

$$\rho \frac{\partial V}{\partial t} + \rho U \frac{\partial V}{\partial x} + \rho V \frac{\partial V}{\partial y} = - \frac{\partial P}{\partial y}$$

These are used throughout the flow field except across the shock where the additional Rankine-Hugoniot relationships are required. Grossman and Moretti (Ref 4) used this choice in their solution technique but this decision resulted in adding appreciable complications when the method was applied to transonic airfoils. The second alternative is to use the steady state form of the same set of equations but written in divergence form because in this form the system is then applicable even across the shock wave. This set still consists of three nonlinear partial differential equations together with an algebraic relationship and therefore remains a very complex system to work with. Although Tai (Ref 11) developed a solution technique based on the subdomain method using this set of governing equations, a simpler system is desirable for this presentation. A third possible choice exists when it is assumed that a good approximation to the flow field can be obtained without mathematically accounting for the shock waves. Spreiter and Alksne

(Ref 10) and several others have used this assumption while working with integral solutions for slender, sharp-nosed bodies in the transonic regime. This choice seems particularly attractive for this presentation since the scope, as stated in the introduction, has already excluded the interaction between the shock and the boundary layer and because, in general, no a priori conclusion can be drawn concerning the magnitude of the change in the flow due exclusively to the inviscid shock effects. Further, since this is a preliminary study of a new solution technique this assumption permits the evaluation of the technique with the minimum mathematical complexity. Then, once the technique is formulated using the simpler set, the same principles could be applied to the more exact equations if the solution required it.

With this decision, small perturbation theory and the irrotationality condition can be used to define a set of governing equations. Liepmann and Roshko (Ref 6:202-208) develop the theory and give the form of the small perturbation relationship which is valid throughout the transonic regime:

$$(1 - M_\infty^2)\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = M_\infty^2 \frac{\gamma + 1}{U_\infty} u \frac{\partial u}{\partial x} \qquad (1)$$

This equation when nondimensionalized and rearranged (Appendix A) may be written

$$\frac{\partial}{\partial x}\left[(1 - M_\infty^2)u - \frac{1}{2} M_\infty^2(\gamma + 1)u^2\right] + \frac{\partial v}{\partial y} = 0 \qquad (2)$$

Traditionally, the nonlinearity of Eq (2) and the change in sign of the first term for some positive value of u as M_∞ approaches one has made this equation difficult to solve. In an attempt to relocate these difficulties into a form which might be more easily dealt with numerically the following definition is made

$$\Omega = (1 - M_\infty^2)u - \frac{1}{2} M_\infty^2(\gamma + 1)u^2 \tag{3}$$

Therefore the full set of governing equations needed to completely define the problem consists of Eq (3), the modified small perturbation relationship,

$$\frac{\partial \Omega}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4}$$

and the condition of irrotationality

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{5}$$

Further, these partial differential equations are subject to boundary conditions on the body surface and at infinity. If the airfoil surface is defined by the relation $y = f(x)$, then the condition at the body surface of no flow normal to the surface (Ref 6:208) may be written

$$\frac{v}{u + U_\infty} = \frac{df}{dx} = f' \tag{6}$$

Because small disturbance theory is being used for the governing equations the only mathematical boundary condition that is specified at infinity is that u and v are both finite. However, at infinity in the

physical problem u and v both are zero and this more restrictive condition is generally assumed whenever there are no regions of supersonic flow extending to the infinity boundary. Therefore for this solution technique the boundary condition which will be imposed at infinity is

$$u = v = \Omega = 0 \tag{7}$$

Based on the difficulty experienced by other authors in applying the boundary conditions, it was decided that a coordinate transformation should be used to avoid assumptions as to where the boundary conditions should be applied. New independent variables were defined as

$$\xi = \frac{x}{|x| + L} \tag{8}$$

$$\eta = \frac{y - f(x)}{|y| + L} \tag{9}$$

where L is some positive constant. As can be seen from Fig. 1 and 2 on the following page, this transformation accomplishes two important objectives. First, the domain of the independent variables is reduced to a finite region; therefore, the boundary conditions at infinity in the x-y plane are applied at unity in the ξ-η plane. Second, the thickness of any airfoil shape is reduced to zero in the new coordinate system such that the boundary condition on the airfoil surface is applied at $\eta = 0$.

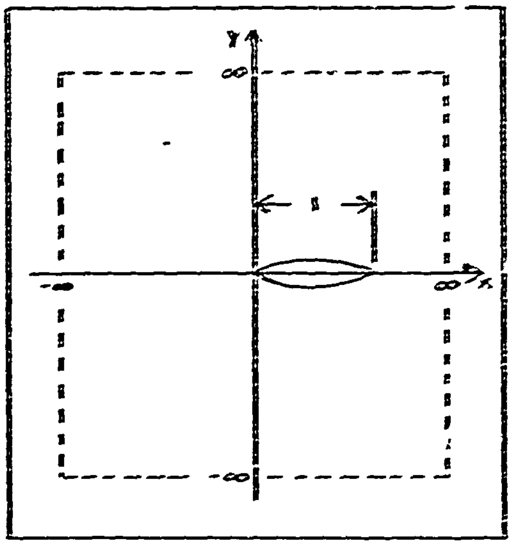

Fig. 1. Original Reference Frame.

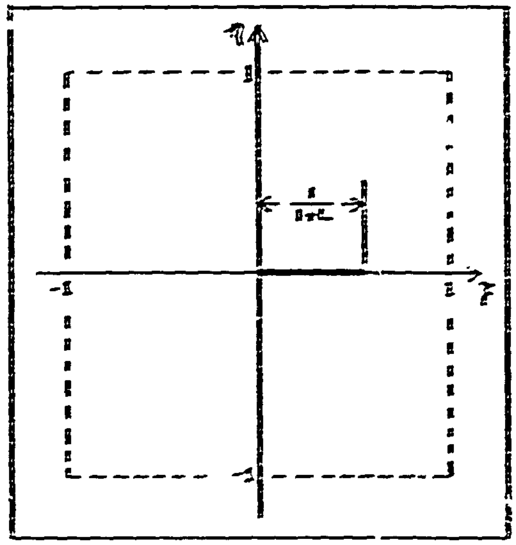

Fig. 2. Transformed Reference Frame.

This transformation is thoroughly discussed and developed in Appendix A where the small perturbation relation is shown to be

$$\frac{\partial\Omega}{\partial\xi} = \frac{Lf'(1-\eta)}{(L+f)(1-|\xi|)^2}\frac{\partial\Omega}{\partial\eta} - \frac{L(1-\eta)^2}{(L+f)(1-|\xi|)^2}\frac{\partial v}{\partial\eta} \tag{10}$$

and the irrotationality condition becomes

$$\frac{\partial v}{\partial\xi} = \frac{Lf'(1-\eta)}{(L+f)(1-|\xi|)^2}\frac{\partial v}{\partial\eta} + \frac{L(1-\eta)^2}{(L+f)(1-|\xi|)^2}\frac{\partial u}{\partial\eta} \tag{11}$$

The algebraic relationship remains unchanged, therefore, Eqs (3), (10), and (11) represent the full set of transformed governing equations. The boundary conditions which these equations are subject to are

$$u = v = \Omega = 0 \tag{12}$$

at ξ or η equal $\pm$ 1 and

$$v = f'(1 + u)$$

at the body surface (η = 0). This last condition is further simplified by recalling that by definition u is much less than one, therefore the boundary condition can be approximated by

$$v = f' \tag{13}$$

at η = 0.

The most difficult decision that had to be made was the choice of a numerical method to solve the resulting set of equations. Because of the objective to reduce computer time and storage space to a minimum and due to the successful applications by Melnik and Ives (Ref 8) and Tai, it was decided very early that a form of the method of weighted residuals should be used. Within this approach several alternatives were tried and a solution technique using the subdomain method was extensively developed. Unfortunately, the lack of any rational procedure for initializing the variables caused the method to be rejected. Thus, although a detailed discussion of this technique is included as Appendix B, this presentation uses the easier to apply method of lines.

The purpose of the method of lines is to reduce a system of partial differential equations to a set of first-order ordinary differential equations. The method of lines presumes that the derivatives in the η-direction can be represented by a finite-difference approximation (details are included in Appendix A). To use this concept, the η-domain is divided by $(N - 1)$ parallel lines, equally spaced between $\eta = 0$ and $\eta = 1$.

$$\eta_i = \frac{i}{N} \; ; \; i = 1, 2, \ldots, (N-1)$$

Representing the derivatives with respect to η by central differences, the governing equations (3), (10), and (11) for the ith line may be written:

$$\Omega_i = (1 - M_\infty^2)u_i - \frac{1}{2} M_\infty^2(\gamma + 1)u_i^2 \qquad (14)$$

$$\frac{d\Omega_i}{d\xi} = \frac{f'(N - i)L}{2(L + f)(1 - |\xi|)^2} (\Omega_{i+1} - \Omega_{i-1}) -$$

$$\frac{L(N - i)^2}{2N(L + f)(1 - |\xi|)^2}(v_{i+1} - v_{i-1}) \qquad (15)$$

$$\frac{dv_i}{d\xi} = \frac{f'(N - i)L}{2(L + f)(1 - |\xi|)^2} (v_{i+1} - v_{i-1}) +$$

$$\frac{L(N - i)^2}{2N(L + f)(1 - |\xi|)^2} (u_{i+1} - u_{i-1}) \qquad (16)$$

When this system is collected for all (N - 1) lines the resulting set of (3N - 3) equations involve (3N + 3) unknowns: u_j, v_j, and Ω_j; $j = 0, 1, 2, \ldots, N$. Three of the six additional relationships which are required to uniquely define the problem are obtained from the boundary condition

$$u_N = v_N = \Omega_N = 0 \tag{17}$$

at $\eta = 1$. Two more relations are obtained by writing governing equations (3) and (10) along the body surface

$$\Omega_0 = (1 - M_\infty^2)u_0 - \frac{1}{2} M_\infty^2(\gamma + 1)u_0^2 \tag{18}$$

$$\frac{d\Omega_0}{d\xi} = \frac{LN}{2(L + f)(1 - |\xi|)^2} [f'(-3\Omega_0 + 4\Omega_1 - \Omega_2) -$$

$$(-3v_0 + 4v_1 - v_2)] \tag{19}$$

where the η-derivatives are approximated by forward-differences. Finally, the last required equation comes from the boundary condition on the surface, Eq (13), which is rewritten as,

$$v_0 = f'(x) = f'\left(\frac{\xi L}{1 - |\xi|}\right) \tag{20}$$

Therefore, the complete problem now consists of 2N first-order ordinary differential equations which must be solved simultaneously with (N + 3) algebraic relations.

This set can be conveniently solved if initial values can be determined for starting the solution of the ordinary

differential equations and if a unique value for u can be found for a given Ω from Eq (3). If Eq (3) is solved for u by the quadratic formula

$$u = \frac{(1 - M_\infty^2)}{M_\infty^2(\gamma + 1)} \quad 1 \pm \sqrt{1 - \frac{2M_\infty^2(\gamma + 1)}{(1 - M_\infty^2)^2}\,\Omega} \qquad (22)$$

then, a choice of signs must be made to uniquely define u. A typical plot of u versus Ω from Eq (3) for a M_∞ less than one is shown below in Fig. 3 where u_c refers to the value of u at which $\frac{\partial\Omega}{\partial u} = 0$; namely,

$$u_c = \frac{(1 - M_\infty^2)}{M_\infty^2(\gamma + 1)} \qquad (23)$$

and Ω_c is the corresponding maximum value of Ω. Comparing Eq (22) and Fig. 1 it can be concluded that the proper sign in Eq (22) is negative if $u \leq u_c$ and positive if $u > u_c$.

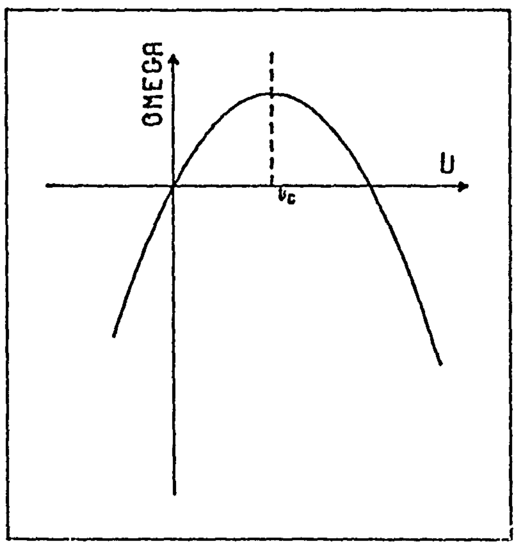

Fig. 3. Typical Relation of u to Ω as found from Eq (22) Including the Critical Values of u.

From experimental data presented by Knechtel (Ref 5) it can be observed that for subcritical flows u is always less than u_c while for supercritical flows u exceeds this value over some portion of the airfoil. Typical examples of the u-distribution on the surface of a subcritical (M_∞ = 0.806) and a supercritical (M_∞ = 0.909) 6% circular arc airfoil are given in Figs. 4 and 5 below. Also shown in these figures are the values of u_c for each case and the corresponding distribution of Ω required when the proper sign is used in Eq (22).

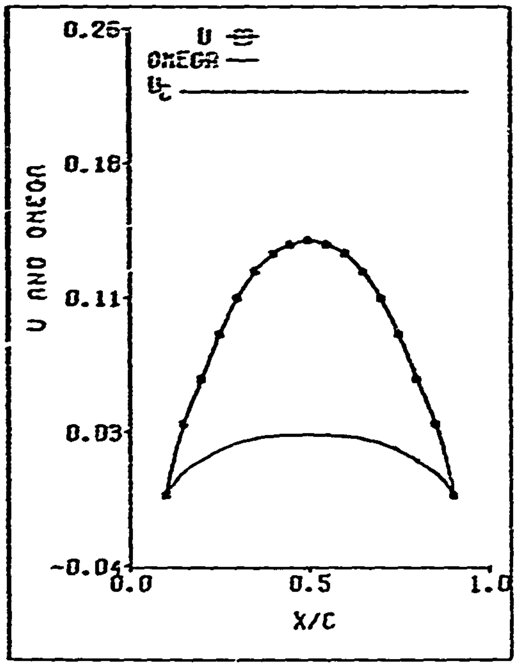

Fig. 4. Subcritical (M_∞ = .806) distribution of u, corresponding Ω-distribution from Eq (22) and critical value of u for a 6% circular arc airfoil.

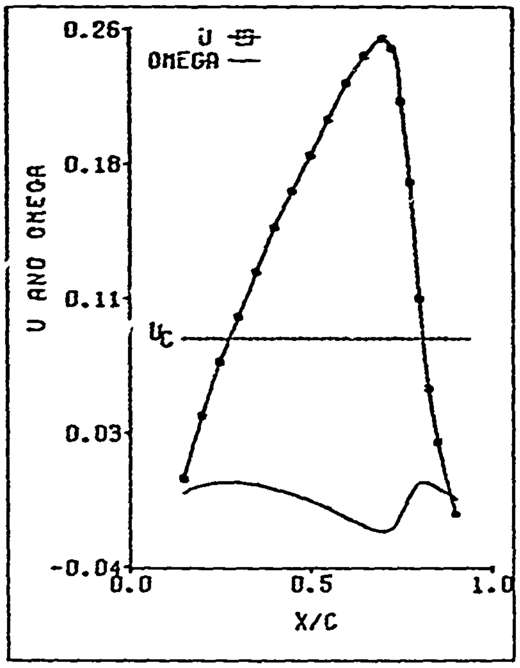

Fig. 5. Supercritical (M_∞ = .909) distribution of u, corresponding Ω-distribution from Eq (22) and critical value of u for a 6% circular arc airfoil.

When the set of governing equations is solved by a marching method such as Runge-Kutta, the proper sign in Eq (22) can be predicted by the value of Ω_i and Ω_i' at the points where the value of u_i is found to equal u_c. Therefore, if the proper initial conditions of v_i and Ω_i can be determined Eq (22) is solved for the corresponding u_i. Then the Runge-Kutta fourth-order solution technique can be applied to solve the 2N ordinary differential equations for the value of v_i and Ω_i some $\Delta\xi$ downstream and thereby setting up the marching technique in the ξ-direction. From this method the values of u and v are found at every point in a $\Delta\xi$ x $\Delta\eta$ grid which covers the entire flow field. With these values three important parameters of the field are found: the pressure distribution over the airfoil surface

$$C_p = -2u_0 \tag{24}$$

the ratio of local to free stream velocity at each grid point

$$\frac{V}{U_\infty}\bigg|_i = \sqrt{(u_i + 1)^2 + v_i^2} \tag{25}$$

and the Mach number distribution throughout the f.eld

$$M_i = \frac{V}{U_\infty} M_\infty\left[1 - \frac{1}{2}(\gamma - 1)(2u_i + u_i^2 + v_i^2)\right]^{-\frac{1}{2}} \tag{26}$$

The solution technique can then be evaluated by a comparison with experimental data.

III. Results

The results and conclusions obtained through the course of this study will be given in chronological order so that the subject matter can be divided up into four main topics: subcritical airfoil studies, supercritical airfoil efforts, attempts to generalize the initial conditions, and applications of the method to subsonic and supersonic wavy walls.

Subcritical Airfoil

It was assumed in the formulation of the solution technique that only symmetrical airfoils at zero incidence angle would be considered. Therefore, because of availability of experimental data accumulated by Knechtel (Ref 5) the 6% circular arc airfoil was selected as the reference airfoil and the subcritical case of $M_\infty = 0.806$ was chosen for the first test of the proposed solution technique.

The difficulties in generating initial conditions for upstream of the airfoil for this problem will be discussed in detail later in this section but a method was constructed at this point for determining the initial conditions by starting the solution at a point on the airfoil ($0 \leq x \leq 1$). Based on the form of the experimental C_p distribution and the curve of the boundary condition $v_0 = f'$ (as shown in Fig. 6 on the next page) it was assumed that the distribution of any $u_i(\xi)$ (recalling that the subscripts on u, v, or Ω refer to their value on the line $\eta = \frac{i}{N}$) is symmetrical with respect to the line $x = 1/2$ while the distribution of any

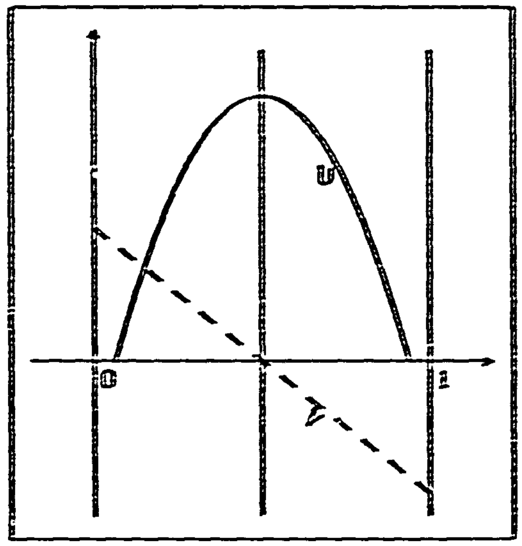

Fig. 6. Typical Distribution of u_0 and v_0 Over a Circular Arc Airfoil.

$v_i(\xi)$ is symmetrical with respect to the point $x = 1/2$, $y = f(1/2)$. With these assumptions the initial values of u_i and v_i can be guessed at a point ahead of $x = 1/2$, then knowing what the values of u_i and v_i should be at some point downstream, these initial guesses are refined by an iteration process. In application, from the experimental data the point at which $C_p = 0$ was selected as the initial value of x, the values of u_i at this point were assumed to be zero, and the values of v_i were guessed. Since the assumed symmetry required all v_i to be zero at $x = 1/2$, these guesses for the initial conditions were refined by iterating between the initial x and $x = 1/2$ until all the v_i changed signs within a region $x = 0.5 \pm 0.002$. When this condition was met it was found that the entire flow field $(0 \leq x \leq 1)$

possessed the symmetry in u_i and v_i as was originally assumed. Having this conformation of the assumptions made on the field, the method was used to generate the initial conditions required by the method of lines solution using 3, 4, 5, 6, and 8 lines. For the subcritical airfoil the resulting C_p distributions are compared to each other in Fig. 7 on the next page. The distribution obtained from the 8-line solution was slightly lower than that obtained with the 6-line solution but they were so close that the experimental data could not be read accurately enough to determine which was the better solution. Consequently, it was concluded that 6-lines were sufficient.

The conclusion which can be drawn from these results is that the method of lines, given the proper initial condition, produces very accurate solutions with low order approximations. Further, since only 12.2 seconds of CDC 6600 computer time was required to generate all the u_i, v_i, M_i, velocity ratios, and C_p distribution for the 6-line case, the solution technique satisfies the time restraint objective required of the method.

Critical and Supercritical Airfoils

With the approach established on the subcritical airfoil the technique was next tested at the critical Mach number for this airfoil, $M_\infty = 0.833$. Using the 6-line approximation the results were found to be in excellent agreement with the experimental data; the C_p distribution was again

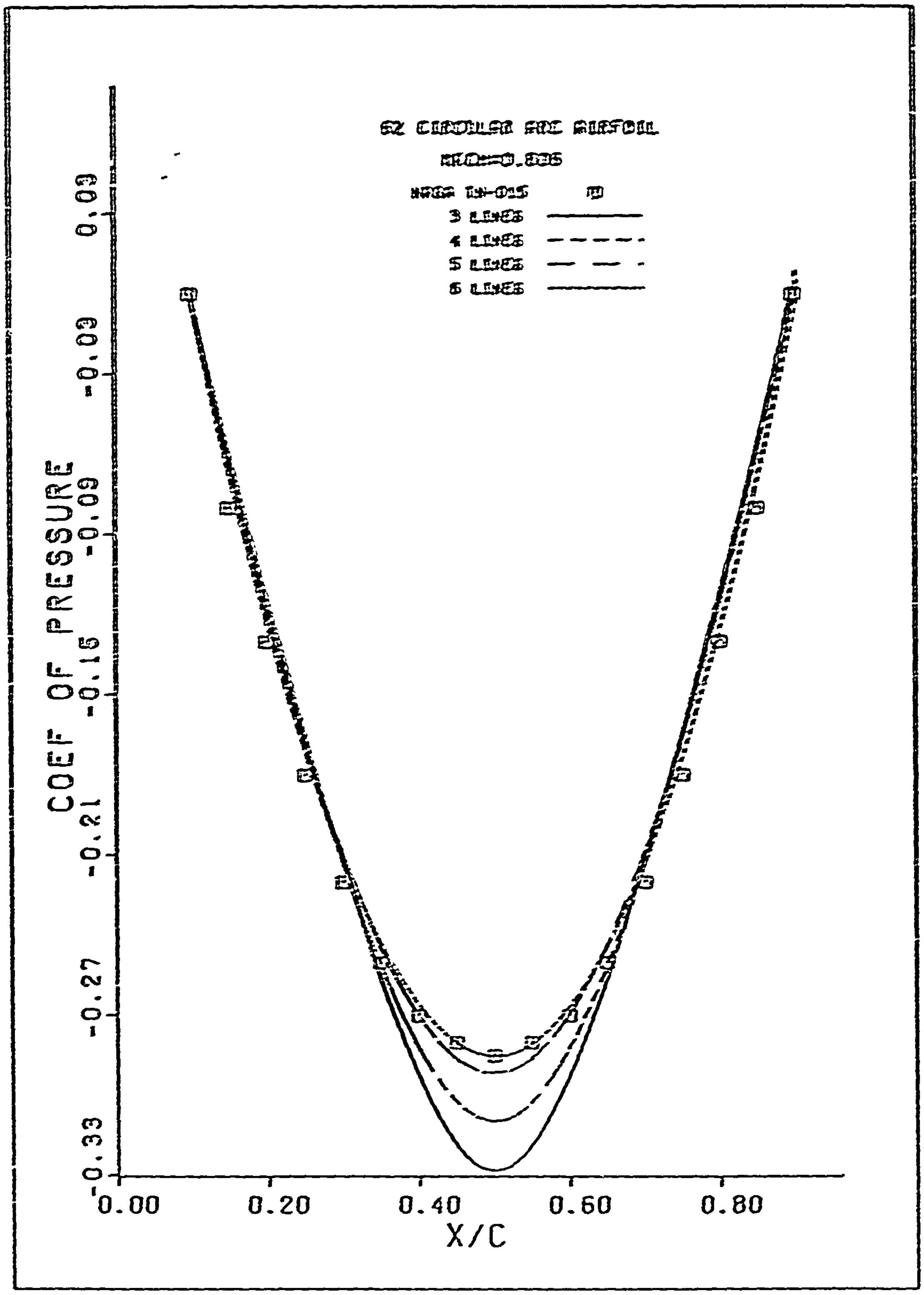

Fig. 7. Comparison of C_p Distributions Obtained with the Proposed Solution Technique versus the Experimentally Determined Distribution for a Subcritical Circular Arc Airfoil.

coincidental to the experimental values and at the grid point $x = 0.502$, $y = 0.0297$ the Mach number was found to be 0.9982 as compared to the experimentally predicted value of $\bar{M}_i = 1$ at the point $x = 0.5$, $y = 0.03$.

For a third test of the solution technique the supercritical case at $M_\infty = 0.85$ was selected because it still appears to meet the symmetry required in predicting the initial conditions. As discussed in Section II, there is a change of signs in the u-Ω relationship given in Eq (22) whenever a sonic line is crossed in the flow field. The computer program was able to determine the points at which this change was to occur and properly predicted the sign but the related requirement for the slope of Ω_i to be zero at the same points (refer to Figs. 4 and 5) introduced an additional complication. Consider the differential equation for Ω_i

$$\frac{\partial \Omega_i}{\partial \xi} = \frac{f'(1 - i/N)L}{(L + f)(1 - |\xi|)^2} \frac{\partial \Omega_i}{\partial \eta} - \frac{(1 - i/N)^2 L}{(L + f)(1 - |\xi|)^2} \frac{\partial v_i}{\partial \eta}$$

Letting ξ^* be the value at which the sign change is to occur, then over a small region surrounding this point, $\xi = \xi^* \pm \varepsilon$ the values of coefficients and $\frac{\partial v_i}{\partial \eta}$ may be considered constant and this expression is simplified to

$$\frac{\partial \Omega_i}{\partial \xi} = C_1 \frac{\partial \Omega_i}{\partial \eta} - C_2$$

Therefore, for a change in sign to occur in $\frac{\partial\Omega_i}{\partial\xi}$ within the region $\xi^* \pm \varepsilon$ it can be seen that a very accurate approximation to $\frac{\partial\Omega_i}{\partial\eta}$ is required. Trying 6 lines in the solution technique (which used a second order approximation for the derivative) failed to be accurate enough to get a change in sign of $\frac{\partial\Omega_i}{\partial\xi}$ when the sign change occurred in the u-Ω relationship which implies that a larger number of lines is needed. This would not pose any serious complication except that the method used to determine initial conditions, although sufficient for a small number of lines, becomes a very tedious and lengthy process if more than eight lines are required. Since there was no way to determine beforehand how many lines would be required to get the accuracy needed and because of the time constraint on the study, the development of this solution had to be set aside in order to work on more generalized problems.

Initial Conditions

By considering only problems with symmetrical airfoils at zero incidence angle and by restricting η to values of $\eta \geq 0$, it will be recalled that the boundary conditions corresponding to the infinity conditions are

$$u_i = v_i = \Omega_i = 0 \text{ if } \xi = \pm 1 \text{ or } \eta = 1 \qquad (27)$$

while at the boundary

$$v_0 = \begin{cases} f' \text{ if } 0 \leq \xi \leq \frac{1}{1 + L} \\ 0 \quad \text{if } \xi < 0 \text{ or } \xi > \frac{1}{1 + L} \end{cases} \qquad (28)$$

Since the solution technique uses a marching method in the ξ-direction and boundary conditions are available at both $\xi = \pm 1$, this suggests that the initial conditions are well established and the solution can progress into difficulty because of two complications which result from relating the boundary and initial conditions to the real physical problem. First, if the solution was started from $\xi = -1$ and the initial conditions were all set equal to zero as predicted by the boundary conditions the set of ordinary differential equations would all equal zero and the technique would predict an undisturbed flow everywhere upstream of the leading edge. Since it is known that the presence of the airfoil does disturb the flow upstream, if the velocity is less than supersonic, it is necessary to start the solution from a point away from the boundary $\xi = -1$ where the initial conditions are not all zero. Second, since in general $f' \neq 0$ at the trailing and leading edge of an airfoil, the boundary condition at $\eta = 0$ causes discontinuities in v_0 at each of these points. If it is attempted to march past these points the jump in value of v_0 is propagated throughout the set of governing equations and the system is found to be unstable. To avoid this instability attempts were made to slightly alter the shape of the airfoil at the leading and trailing edges such that v_0 changed from 0 to f' over a small region rather than at a single point.

In developing this theory for predicting the initial condition smoothing functions of the type

$$v_0 = f'(\varepsilon)\,\frac{x}{\varepsilon}\,;\ 0 \le x \le \varepsilon$$

$$v_0 = f'(0)\left(1 + \frac{x}{\varepsilon}\right);\ -\varepsilon \le x \le 0$$

$$v_0 = f'(\varepsilon)\left(\frac{1}{2} + \frac{x}{2\varepsilon}\right)\ ;\ -\varepsilon \le x \le \varepsilon$$

$$v_0 = \frac{1}{2}\,f'(\varepsilon)\left[1 + \cos \pi\left(1 + \frac{x}{\varepsilon}\right)\right];\ 0 \le x \le \varepsilon$$

were tested at the leading edge while the initial conditions at $\xi = -0.9$ were perturbed in a variety of combinations and magnitudes. With this data it was hoped to correlate the change in initial conditions to the change in the solution over the airfoil but no combination of initial conditions and smoothing functions could be found to give stable solutions past the leading edge. The solution over the subcritical airfoil was produced by the method described previously at this time and with this 4-line solution in conjunction with the forcing function

$$v_0 = f'(0.9)[(1.0 - x)/0.1]$$

a concentrated effort was made to march off the back of the airfoil and correlate the change in initial conditions at $x = 1/2$ to the flow field that resulted at $x = 2$. It was found that a perturbation on the order of 10^{-4} imposed on any one of the initial conditions would substantially alter the flow at $x = 2$ and that although the magnitude of all the u_i and v_i decreased downstream of the trailing edge they eventually would diverge and no set of initial conditions

could be found to force convergence to undisturbed flow far downstream. Because of the sensitivity of the flow field to the initial conditions, the value of u_i and v_i marching on or off both ends of the smoothing functions are critical to the solution. Consequently, no way could be found to separate the influence of a change in initial conditions from the error injected by the smoothing functions and without this separation no rationale could be found for the selection of either one.

Wavy Wall Solutions

It was realized very late in the study that the solution technique could be applied equally well to an infinite wavy wall. The exact solutions for C_p, u, and v for both the subsonic and supersonic case are developed by Liepmann and Roshko (Ref 6) and offer a very efficient means of testing the proposed solution technique. Although these exact solutions cannot be used to determine the applicability of the method to transonic problems, they can serve to evaluate the accuracy of the solution technique in predicting the flow field as given by the exact solutions if the method is started with the correct initial conditions.

To perform this evaluation the system of equations were first reduced to the simplified problem solved by Liepmann and Roshko by approximating the Ω-u relationship given in Eq (22) as

$$\Omega = (1 - M_\infty^2)u$$

and by simplifying the y-η coordinate transformation to

$$\eta = \frac{y}{y + L}$$

so the boundary condition at $y = f(x)$ could be approximately applied at $y = 0$. The wall function, $f(x)$, was defined by

$$f(x) = 0.01 \sin 2\pi x$$

where the wave amplitude was selected as 0.01 with a wave length of 1. By choosing $x = 0$ as the starting point the initial conditions were defined as

$$u_i = 0$$

$$v_i = (0.01)(2\pi)e^{-\frac{Li}{N-i}(2\pi)(1-M_\infty)^{1/2}}$$

$$i = 0, 1, 2, \ldots, (N-1)$$

for the subsonic case and

$$u_i = -(0.01)(2\pi)(M_\infty{}^2-1)^{-1/2} \cos\left[-2\pi \frac{Li}{N-i} M_\infty{}^2-1)^{1/2}\right]$$

$$v_i = (0.01)(2\pi) \cos\left[-2\pi \frac{Li}{N-i} (M_\infty{}^2-1)^{1/2}\right]$$

$$i = 0, 1, 2, \ldots, (N-1)$$

for the supersonic case where N is the number of lines to be used in the solution. Finally, in order to test the technique on both cases, $M_\infty = 0.2$ and $M_\infty = 2.0$ were arbitrarily selected as the subsonic and supersonic Mach numbers.

Because finite-differences are used to approximate the derivatives and since u and v are changing most rapidly in the vicinity of the wall, it is necessary to have a good distribution of lines in this region if the solution is to be accurate. Solving Eq (29) for y and letting $\eta = i/N$ implies that

$$y_i = \frac{iL}{N - i}$$

from which it can be observed that the value of L determines how close the ith line is to the body. With a large number of lines ($N > 15$) many of the lines are naturally close to the body due to the coordinate transformation and a change in L ($0 < L < 3$) has little effect on the solution. For a smaller number of lines L becomes increasingly important and for less than 8 lines a substantial change in the solution was observed as L was varied between 0 and 2. Figure 8 on the page following this section shows the 4-line solution for the subsonic wavy wall with $L = 0.5$ while Fig. 9 shows the improvement in this solution as L is changed to 0.65. In general, an optimum value of L could be found for every value of N but in each case tested $L = 0.5$ gave reasonable results. Therefore, $L = 0.5$ was selected as the reference value of L used in all solutions.

As was pointed out in the formulation of this solution technique, the boundary condition at infinite was defined to be

$$u_i = v_i = \Omega_i = 0 \quad @ \quad |\xi| = 1 \text{ or } \eta = 1$$

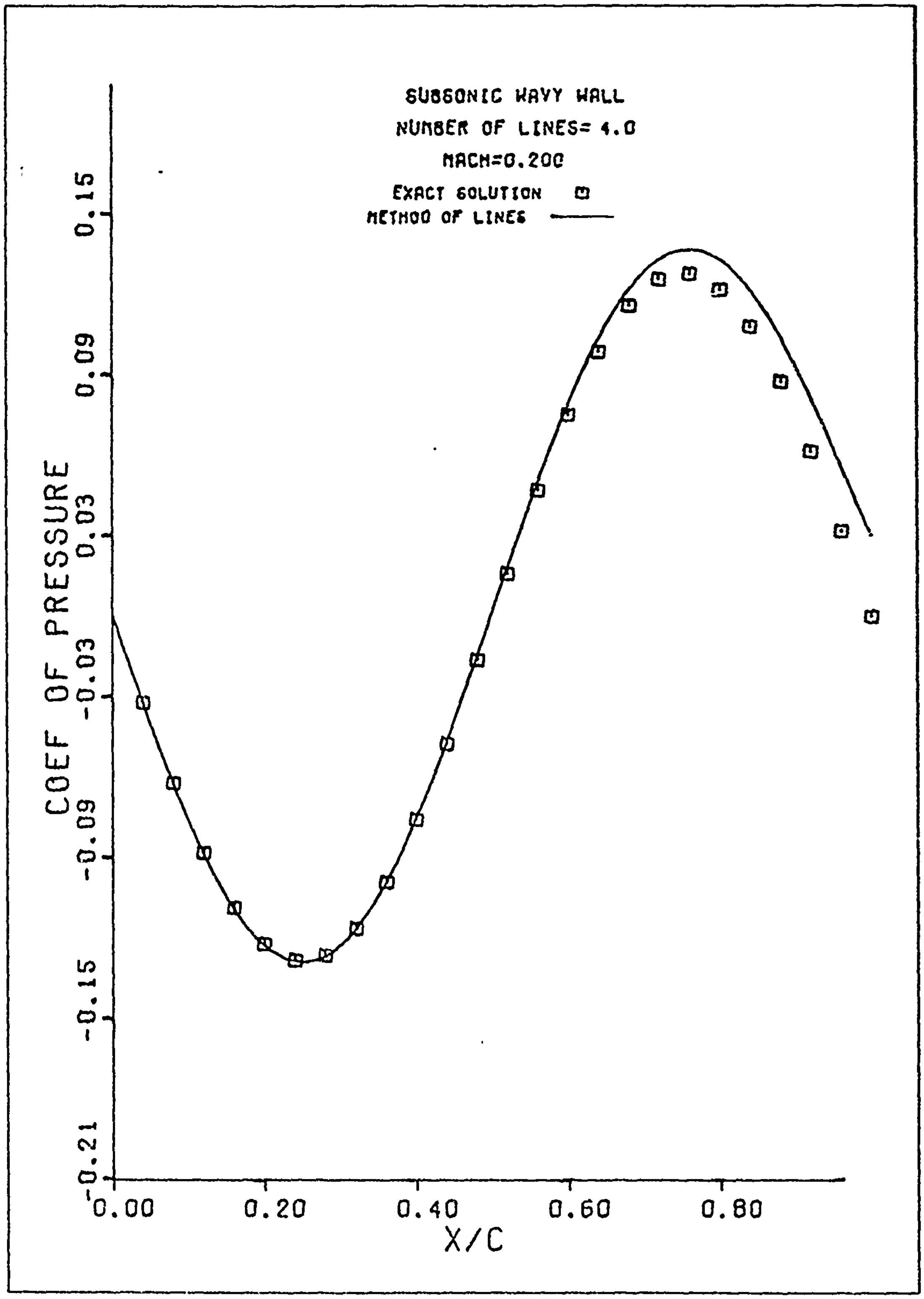

Fig. 8. Solution Obtained for the C_p Distribution over a Subsonic Wavy Wall Using 4 Lines with L = 0.5 as Compared to the Exact Solution.

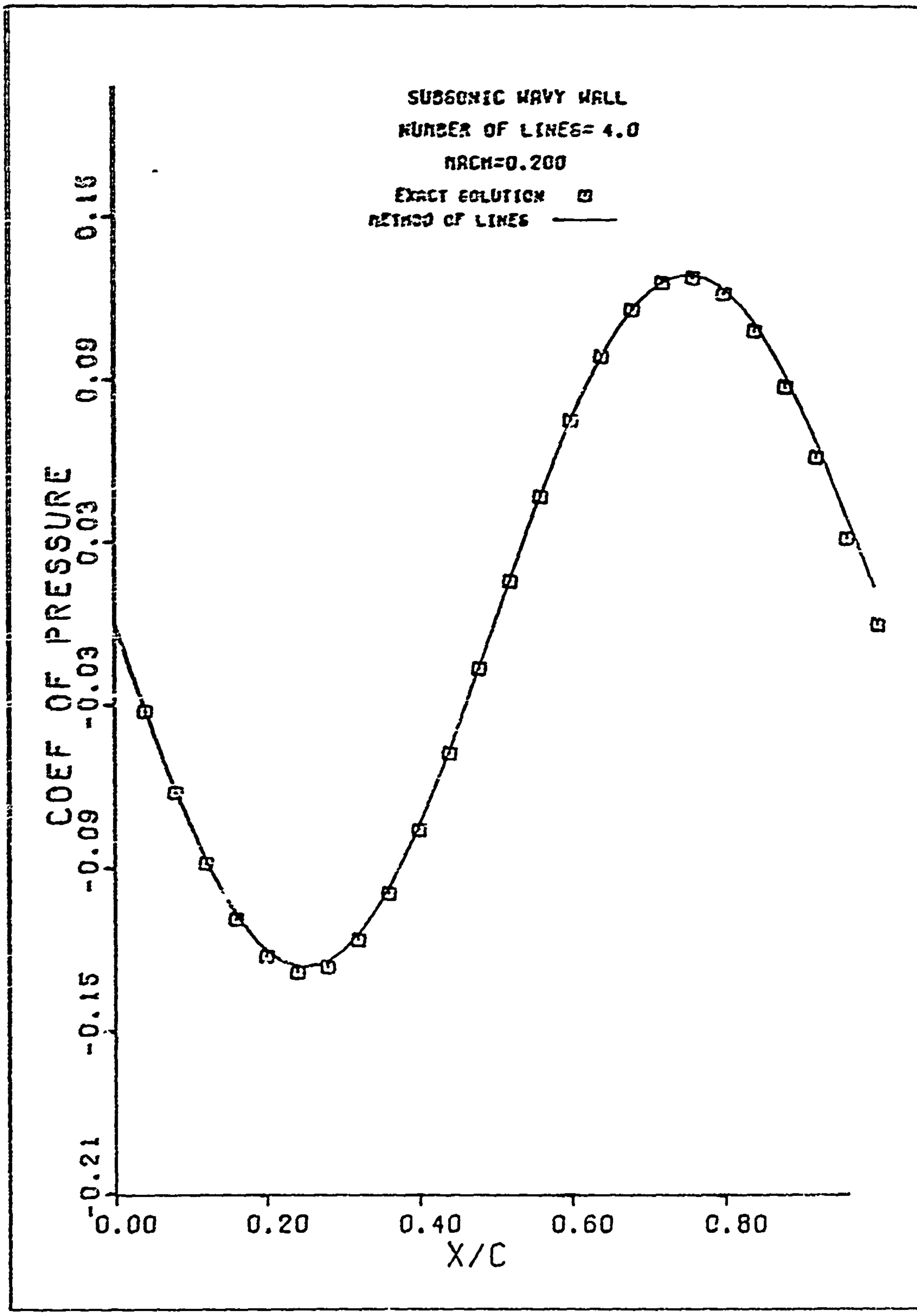

Fig. 9. Solution Obtained for the C_p Distribution over a Subsonic Wavy Wall Using 4 Lines with $L = 0.65$ as Compared to the Exact Solution.

For the subsonic wavy wall the exact solution agreed with this condition and consequently the solution required only 4-lines to get the accuracy as shown in Fig. 9. However, with the supersonic wavy wall the exact solutions predicts that u and v vary sinusoidally at infinity and although this solution gives the range of these values it can not be used to predict specific values for u or v at particular points on the boundary. When u, v, and Ω were set to zero at $\eta = 1$ errors resulted in the solution. However, as the number of lines was increased the solution converged rapidly and in Fig. 10 the C_p distribution obtained using 20 lines is shown to compare very well with the exact solution.

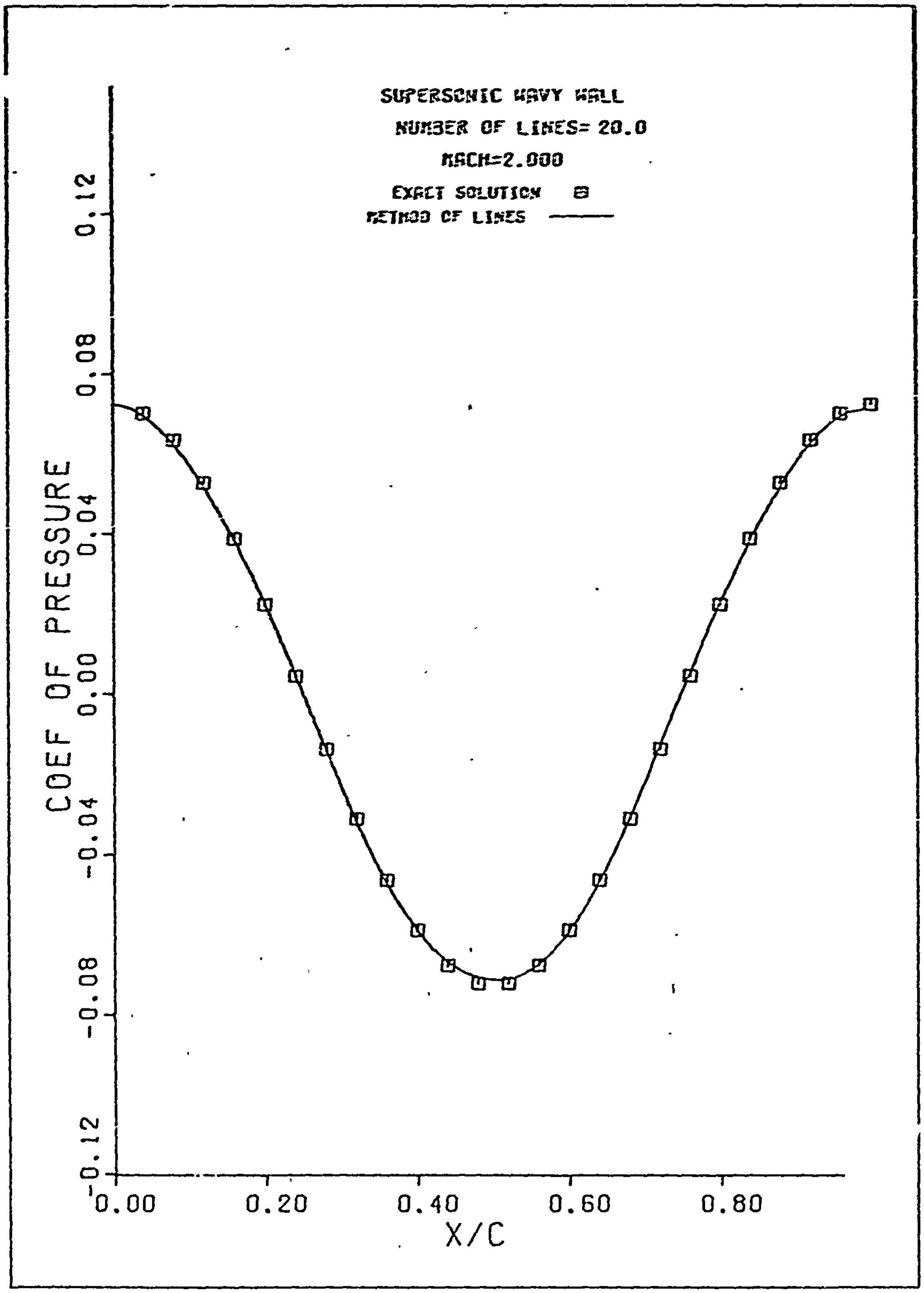

Fig. 10. Solution Obtained for the C_p Distribution over a Supersonic Wavy Wall Using 20 Lines with L = 0.5 as Compared to the Exact Solution.

IV. Conclusions and Recommendations

Conclusions

From the results obtained for the subsonic and supersonic wavy wall and the solutions found for the circular arc airfoil it can be concluded that the proposed solution technique is capable of producing extremely accurate results with very short computer times provided it is furnished with correct initial conditions.

Recommendations

The following recommendations are proposed for future work:

1. It is suggested that for this problem an unsteady system of governing equations should be used. With this three-dimensional coordinate system the marching method can be set up in the time direction and a method of weighted residuals could be used to approximate the derivatives in the x and y directions. In this form, the problem of determining generalized initial conditions can be separated from the problem of handling the discontinuities in the x-y plane and their individual effects can be evaluated and controlled.

2. A coordinate transformation of the form used in this development should be used to control the domain of the independent variables. Although the opportunity was not available to demonstrate the advantage of shrinking the airfoil thickness to zero, it is suggested that this benefit

be retained because of the thick airfoils used in the transonic regime.

3. The introduction of Ω to separate the nonlinearity from the differential equation made the system much easier to work with and it is suggested that this same technique should be applied whenever similar nonlinear terms appear in the differential relations.

4. The method of lines, as used in this development, is highly recommended as a technique for reducing the partial differential equations. Besides being a very straightforward method to apply it has an outstanding advantage in that the resulting unknowns can be easily identified with the physical problem.

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Appendix A

Details of the Solution Technique Formulation

This appendix serves as an extension to the development as presented in Section II of the main body. Its purpose is to give a detailed discussion and formulation of the important mathematical steps in areas where details of the solution technique might be desired by some readers. Three primary subjects will be covered: (1) the nondimensionalization of the basic governing equations, (2) the coordinate transformation, and (3) the application of the method of lines.

Nondimensionalization

The set of governing equations which were selected for this presentation are the small perturbation relationship

$$(1 - M_\infty^2) \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = M_\infty^2 \frac{\gamma+1}{U_\infty} u \frac{\partial u}{\partial x}$$

which may be written by rearranging the derivatives as

$$\frac{\partial}{\partial x} [(1 - M_\infty^2)u - \frac{1}{2} M_\infty^2 \frac{\gamma+1}{U_\infty} u^2] + \frac{\partial v}{\partial y} = 0 \qquad (29)$$

plus the irrotationality condition

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad (30)$$

In these equations M_∞ and U_∞ are the free stream Mach number and uniform velocity, γ is the ratio of specific heats, and

u and v are the perturbations which are defined by

$$\bar{U} = (U_\infty + u)\hat{e}_x + v\hat{e}_y$$

where $\bar{U}$ is the local total velocity vector. These differential governing equations are subject to the boundary conditions at x or y equals infinity where

$$u = v = 0 \tag{31}$$

and on the surface of the body where

$$\frac{v}{u + U_\infty} = \frac{df}{dx} = f' \tag{32}$$

and $y = f(x)$ defines the shape of the airfoil.

To generalize these relationships it is desirable to express them in their nondimensional form. To do so the lengths are nondimensionalized with respect to the chord length

$$x^* = \frac{x}{c}$$

$$y^* = \frac{y}{c}$$

$$f^* = \frac{f}{c}$$

and the velocities are nondimensionalized with respect to the free stream velocity

$$u^* = \frac{u}{U_\infty}$$

$$v^* = \frac{v}{U_\infty}$$

Making these definitions and substituting into Eqs (29) through (32) changes the form of these expressions to

$$\frac{\partial}{\partial x^*}\left[(1 - M_\infty^2)u^* - \frac{1}{2} M_\infty^2(\gamma+1)u^{*2}\right] + \frac{\partial v^*}{\partial y^*} = 0 \qquad (33)$$

$$\frac{\partial v^*}{\partial x^*} - \frac{\partial u^*}{\partial y^*} = 0 \qquad (34)$$

$$u^* = v^* = 0 \quad @ \infty \qquad (35)$$

$$\frac{v^*}{u^* + 1} = \frac{df^*}{dx^*} \quad @\ y^* = f^* \qquad (36)$$

Dropping the superscripts for simplicity of notation gives the equations as they are used in the main body.

<u>Coordinate Transformation</u>

It was decided to use a coordinate transformation because it is desirable to limit the domain of the independent variables. The specific transformation chosen to accomplish this objective is;

$$\xi = \frac{x}{|x| + L} \qquad (37)$$

$$\eta = \frac{y - f(x)}{|y| + L} \qquad (38)$$

where L is some positive constant. The relationship between the original independent variables and the new coordinates is shown graphically in Figures A1 and A2. As can be seen from these figures, as the $|x|$ or $|y|$ go to infinity the $|\xi|$ and

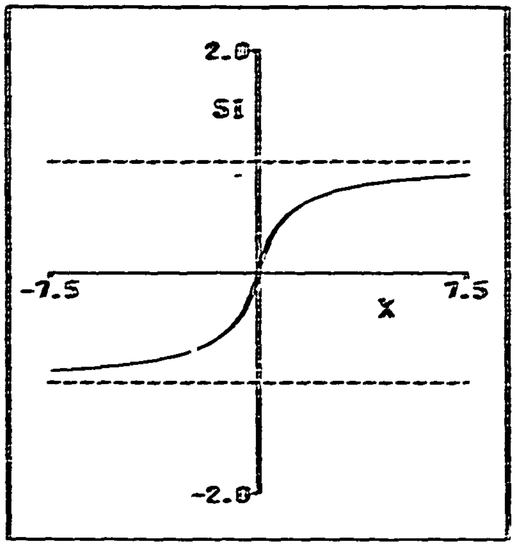

Fig. A1. Relationship between the Independent Variables $x - \xi$.

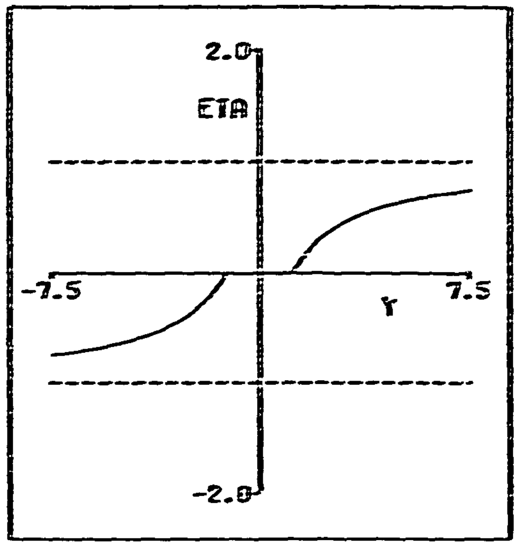

Fig. A2. Relationship between the Independent Variables $y - \eta$.

$|\eta|$ are restricted to values of less than one therefore these transformations reduce the domain of the independent variables to finite regions. Also Fig. A2 shows the additional benefit of reducing the thickness of any airfoil in the x-y plane to zero in the ξ-η plane but it should be noted that the plot of y-η is dependent on the value of x because f(x) appears in Eq (38).

In order to transform the governing equations into the new coordinate system the partial derivatives must be expressed in terms of ξ and η. If some dependent variable is defined by

$$g(x,y) = G(\xi,\eta)$$

then the partial derivatives with respect to x and y may be written by the chain rule as

$$\frac{\partial g}{\partial x} = \frac{\partial G}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial G}{\partial \eta}\frac{\partial \eta}{\partial x} \tag{39}$$

$$\frac{\partial g}{\partial y} = \frac{\partial G}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial G}{\partial \eta}\frac{\partial \eta}{\partial y} \tag{40}$$

From Eqs (37) and (38) the partials of ξ and η with respect to x and y are evaluated to be

$$\frac{\partial \xi}{\partial x} = \frac{L}{(|x| + L)^2}$$

$$\frac{\partial \xi}{\partial y} = 0$$

$$\frac{\partial \eta}{\partial x} = \frac{-f'}{|y| + L}$$

$$\frac{\partial \eta}{\partial y} = \frac{L + |f|}{(|y| + L)^2}$$

or in terms of the new independent variables

$$\frac{\partial \xi}{\partial x} = \frac{(1 - |\xi|)^2}{L}$$

$$\frac{\partial \eta}{\partial x} = \frac{-f'(1 - |\eta|)}{L + |f|}$$

$$\frac{\partial \eta}{\partial y} = \frac{(1 - |\eta|)^2}{L + |f|}$$

For flow fields symmetrical about $\eta = 0$, only the upper half plane is considered and Eqs (39) and (40) may be written as

$$\frac{\partial g}{\partial x} = \frac{(1 - |\xi|)^2}{L} \frac{\partial G}{\partial \xi} - \frac{f'(1-\eta)}{L+f} \frac{\partial G}{\partial \eta} \tag{41}$$

$$\frac{\partial g}{\partial y} = \frac{(1-\eta)^2}{L+f} \frac{\partial G}{\partial \eta} \tag{42}$$

Returning to the differential governing equations as given in Eqs (33) and (34) and defining

$$\Omega = (1 - M_\infty{}^2)u - \frac{1}{2} M_\infty{}^2(\gamma+1)u^2 \tag{43}$$

then with the definitions given in Eqs (41) and (42) the governing equations in the new coordinate system are

$$\frac{(1 - |\xi|)^2}{L} \frac{\partial \Omega}{\partial \xi} - \frac{f'(1-\eta)}{L+f} \frac{\partial \Omega}{\partial \eta} + \frac{(1-\eta)^2}{L+f} \frac{\partial v}{\partial \eta} = 0$$

$$\frac{(1 - |\xi|)^2}{L} \frac{\partial v}{\partial \xi} - \frac{f'(1-\eta)}{L+f} \frac{\partial v}{\partial \eta} - \frac{(1-\eta)^2}{L+f} \frac{\partial u}{\partial \eta} = 0$$

To get the form of these equations desired for application of the method of lines solution technique, derivatives with respect to ξ remain on the left hand side such that

$$\frac{\partial \Omega}{\partial \xi} = \frac{f'(1-\eta)L}{(L+f)(1 - |\xi|)^2} \frac{\partial \Omega}{\partial \eta} - \frac{(1-\eta)^2 L}{(L+f)(1 - |\xi|)^2} \frac{\partial v}{\partial \eta} \tag{44}$$

$$\frac{\partial v}{\partial \xi} = \frac{f'(1-\eta)L}{(L+f)(1 - |\xi|)^2} \frac{\partial v}{\partial \eta} + \frac{(1-\eta)^2 L}{(L+f)(1 - |\xi|)^2} \frac{\partial u}{\partial \eta} \tag{45}$$

Finally, the boundary conditions as given in Eqs (35) and (36) become

$$u = v = \Omega = 0 \quad @|\xi| \text{or } \eta = 1$$

and

$$\frac{v}{u+1} = f' \quad @ \; \eta = 0$$

in the new coordinate system.

Applying the Method of Lines

In general, the functions u and v are changing much more rapidly in the ξ-direction than in the η-direction. Therefore, to use the method of lines the derivatives with respect to η are chosen to be represented by a finite-difference approximation. Along any line parallel to the η-axis, except for $\eta = 0$, the first-order partial of a function $G(\xi,\eta)$ with respect to η is approximated by

$$\frac{\partial G}{\partial \eta} \approx \frac{G(\xi,\eta + \Delta\eta) - G(\xi,\eta - \Delta\eta)}{2\Delta\eta} \tag{46}$$

which is the second-order central-difference. For the line $\eta = 0$, since only values of η greater than zero are considered, the partial is approximated by the second-order forward difference

$$\left.\frac{\partial G}{\partial \eta}\right|_{\eta=0} \approx \frac{1}{2\Delta\eta} \left[-3G(\xi,0) + 4G(\xi,\Delta\eta) - G(\xi,2\Delta\eta)\right] \tag{47}$$

where in both Eq (46) and (47) the $\Delta\eta$ refers to the distance between evenly spaced lines. Dividng the η-domain by (N-1) of these parallel lines between $\eta = 0$ and $\eta = 1$,

$$\eta = 0, \frac{1}{N}, \frac{2}{N}, \ldots, 1$$

then the value of η along the ith line is written

$$\eta_i = \frac{i}{N} \tag{48}$$

Introducing subscripts corresponding to the value of η and using Eq (48) implies that the partials in Eqs (46) and (47) may be written

$$\frac{\partial G}{\partial \eta} = \frac{N}{2}\,[G_{i+1} - G_{i-1}] \tag{49}$$

$$\frac{\partial G_0}{\partial \eta} = \frac{N}{2}\,[-3G_0 + 4G_1 - G_2] \tag{50}$$

Applying these definitions in the governing equations as given in Eqs (44) and (45) reduces these partial differential relationships to a set of first-order ordinary differential equations of the form

$$\frac{d\Omega_i}{d\xi} = \frac{f'(1 - i/N)L}{(L+f)(1 - |\xi|)^2}\,\frac{N}{2}\,(\Omega_{i+1} - \Omega_{i-1}) - \frac{(1 - i/N)^2 L}{(L+f)(1 - |\xi|)^2}\,\frac{N}{2}\,(v_{i+1} - v_{i-1})$$

$$\frac{d\Omega_o}{d\xi} = \frac{f'NL}{2(L+f)(1 - |\xi|)^2}\,(-3\Omega_0 + 4\Omega_1 - \Omega_2) - \frac{LN}{2(L+f)(1 - |\xi|)^2}\,(-3v_0 + 4v_1 - v_2)$$

$$\frac{dv_i}{d\xi} = \frac{f'(1 - i/N)L}{(L+f)(1 - |\xi|)^2}\,\frac{N}{2}\,(v_{i+1} - v_{i-1}) +$$

$$\frac{(1 - i/N)^2 L}{(L+f)(1 - |\xi|)^2}\,\frac{N}{2}\,(u_{i+1} - u_{i-1})$$

$$\frac{dv_o}{d\xi} = \frac{f'NL}{2(L+f)(1 - |\xi|)^2}\,(-3v_0 + 4v_1 - v_2) +$$

$$\frac{LN}{2(L+f)(1 - |\xi|)^2}\,(-3u_0 + 4u_1 - u_2)$$

Appendix B

A Solution Technique Based on the Method of Weighted Residuals

This appendix presents an alternate solution technique for the transonic airfoil problem. This is the original approach developed for this study but had to be abandoned when no rationale procedure for initializing the variables could be found. Since the solution technique appears powerful enough to warrant further investigation if the initialization problem can be overcome, a complete formulation of the theory is given here.

The set of governing equations used by this technique are the same as those used with the method of lines. After nondimensionalizing and applying the coordinate transformation the set consists of

$$\frac{\partial \Omega}{\partial \xi} = \frac{Lf'(1-\eta)}{(L+f)(1-|\xi|)^2}\frac{\partial \Omega}{\partial \eta} - \frac{L(1-\eta)^2}{(L+f)(1-|\xi|)^2}\frac{\partial v}{\partial \eta} \tag{51}$$

$$\frac{\partial v}{\partial \xi} = \frac{Lf'(1-\eta)}{(L+f)(1-|\xi|)^2}\frac{\partial v}{\partial \eta} + \frac{L(1-\eta)^2}{(L+f)(1-|\xi|)^2}\frac{\partial u}{\partial \eta} \tag{52}$$

$$u = \frac{(1-M_\infty^2)}{M_\infty^2(\gamma+1)} \quad 1 \pm \sqrt{1 - \frac{2M_\infty^2(\gamma+1)}{(1-M_\infty^2)^2}\,\Omega} \tag{53}$$

The boundary conditions to be satisfied are: at the body surface, $\eta = 0$,

$$\frac{\partial v}{\partial \xi} = \frac{L}{(1 - |\xi|)^2} f'' \tag{54}$$

and at infinity, ξ or $\eta = \pm 1$,

$$u = v = \Omega = 0 \tag{55}$$

For this formulation the dependent variables u, v, and Ω are approximated by a series expansion in η where the terms are arranged such that the series automatically satisfy the boundary condition at $\eta = 1$

$$u(\xi,\eta) = \sum_{i=1}^{M} a_i(\xi)(1-\eta)^i \tag{56}$$

$$v(\xi,\eta) = \sum_{j=1}^{N} b_j(\xi)(1-\eta)^j \tag{57}$$

$$\Omega(\xi,\eta) = \sum_{k=1}^{P} c_k(\xi)(1-\eta)^k \tag{58}$$

Substituting these into Eqs (49) through (52)

$$\sum_{k=1}^{P} c_k'(1-\eta)^k = \frac{-L}{(L + f)(1 - |\xi|)^2}$$

$$\left[f' \sum_{k=1}^{P} c_k k(1-\eta)^k - \sum_{j=1}^{N} b_j j(1-\eta)^{j+1} \right] \tag{59}$$

$$\sum_{j=1}^{N} b_j'(1-\eta)^j = \frac{-L}{(L + f)(1 - |\xi|)^2}$$

$$\left[f' \sum_{j=1}^{N} b_j j(1-\eta)^j + \sum_{i=1}^{M} a_i i(1-\eta)^{i+1} \right] \quad (60)$$

$$\sum_{i=1}^{M} a_i(1-\eta)^i = \frac{(1 - M_\infty^2)}{M_\infty^2(\gamma + 1)}$$

$$\left[1 \pm \sqrt{1 - \frac{2M_\infty^2(\gamma + 1)}{(1 - M_\infty^2)^2} \sum_{k=1}^{P} c_k(1-\eta)^k} \right] \quad (61)$$

$$\sum_{j=1}^{N} b_j' = \frac{L}{(1 - |\xi|)^2} f'' \quad (62)$$

In this set of equations at a given η there are a total of $(M + N + P)$ unknowns; namely, all the coefficients a_i, b_j, and c_k. Therefore, a unique solution for this sytem requires that the same number of independent equations be derived from Eqs (59) through (62). The method of weighted residuals offers many techniques by which this system of equations may be developed (Ref 3). Of these techniques, the collocation method is the easiest to apply and was selected for use here. Basically, this method assumes that an approximate solution for the coefficients a_i, b_j, and c_k can be obtained if the Eqs (59) through (62) are satisfied at a finite number of η-values. The collocation method allows the user to select the number

of each type of equation and the particular values of η to be used in obtaining these equations. In this application the equations were selected in the following manner. By allowing η to take on the values

$$\eta = \frac{p-1}{P} \; ; \; p = 1, 2, \ldots, P$$

in Eq (59), P ordinary differential equations are obtained involving the c_k' terms. In Eq (60), η is defined to be

$$\eta = \frac{n}{N} \; ; \; n = 1, 2, \ldots, (N-1)$$

which together with the boundary condition in Eq (62) gives N ordinary differential equations involving the b_j' terms. Finally, M algebraic relationships are obtained from Eq (61) by letting η take on the values

$$\eta = \frac{m-1}{M} \; ; \; m = 1, 2, \ldots, M$$

Collecting this set gives the complete new system of governing equations

$$\sum_{k=1}^{P} c_k' \left(1 - \frac{p-1}{P}\right)^k = \frac{-L}{(L + f)(1 - |\xi|)^2}$$

$$\left[f' \sum_{k=1}^{P} c_k k \left(1 - \frac{p-1}{P}\right)^k + \sum_{j=1}^{N} b_j j \left(1 - \frac{p-1}{P}\right)^{j+1}\right] \quad (63)$$

$$\sum_{j=1}^{N} b_j' \left(1 - \frac{n}{N}\right)^j = \frac{-L}{(L + f)(1 - |\xi|)^2}$$

$$\left[f' \sum_{j=1}^{N} b_j j \left(1 - \frac{n}{N}\right)^j + \sum_{i=1}^{N} a_i i \left(1 - \frac{n}{N}\right)^{i+1} \right] \quad (64)$$

$$\sum_{j=1}^{N} b_j' = \frac{L}{(1 - |\xi|)^2} f'' \quad (65)$$

$$\sum_{i=1}^{M} a_i \left(1 - \frac{m-1}{M}\right)^i = \frac{(1 - M_\infty^2)}{M_\infty^2(\gamma + 1)}$$

$$\left[1 \pm \sqrt{1 - \frac{2M_\infty^2(\gamma + 1)}{(1 - M_\infty^2)^2} \sum_{k=1}^{P} c_k \left(1 - \frac{m-1}{M}\right)^k} \right] \quad (66)$$

The sign choice in Eq (66) is made in the same manner as discussed in Section II of the main body where the critical value of u for this case is

$$u_c = \left[\sum_{i=1}^{M} a_i \left(1 - \frac{m-1}{M}\right)^j \right]_c = \frac{(1 - M_\infty^2)}{M_\infty^2(\gamma + 1)}$$

To solve this system observe that if initial conditions are known for the coefficients b_j and c_k then Eq (66) can be solved for the corresponding values of a_i. Everything is then known on the right sides of Eqs (63) through (65) and the system of ordinary differential equations can be solved

by the Runge-Kutta method to set up the marching method used throughout the field. Unfortunately, no rationale could be found for determining the initial conditions needed to start this solution technique. With the method of lines the dependent variables to be determined were the perturbations at various values of η (u_i and v_i). A knowledge of typical distributions of u and v then could be used to determine the required initial condition when the solution was started on the airfoil surface. With this collocation technique though, the variables to be solved for are the coefficients of series (a_i, b_j, and c_k) and since it was unknown how a typical distribution of these terms might look, it was not possible to use a similar rationale for determining the initial conditions.

Vita

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